

# Polar actions on compact Euclidean hypersurfaces.

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**Abstract:** Given an isometric immersion  $f: M^n \rightarrow \mathbb{R}^{n+1}$  of a compact Riemannian manifold of dimension  $n \geq 3$  into Euclidean space of dimension  $n + 1$ , we prove that the identity component  $Iso^0(M^n)$  of the isometry group  $Iso(M^n)$  of  $M^n$  admits an orthogonal representation  $\Phi: Iso^0(M^n) \rightarrow SO(n + 1)$  such that  $f \circ g = \Phi(g) \circ f$  for every  $g \in Iso^0(M^n)$ . If  $G$  is a closed connected subgroup of  $Iso(M^n)$  acting locally polarly on  $M^n$ , we prove that  $\Phi(G)$  acts polarly on  $\mathbb{R}^{n+1}$ , and we obtain that  $f(M^n)$  is given as  $\Phi(G)(L)$ , where  $L$  is a hypersurface of a section which is invariant under the Weyl group of the  $\Phi(G)$ -action. We also find several sufficient conditions for such an  $f$  to be a rotation hypersurface. Finally, we show that compact Euclidean rotation hypersurfaces of dimension  $n \geq 3$  are characterized by their underlying warped product structure.

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**Key words:** *polar actions, rotation hypersurfaces, isoparametric submanifolds, rigidity of hypersurfaces, warped products.*

## 1 Introduction

Let  $G$  be a connected subgroup of the isometry group  $Iso(M^n)$  of a compact Riemannian manifold  $M^n$  of dimension  $n \geq 3$ , which we always assume to be connected. Given an isometric immersion  $f: M^n \rightarrow \mathbb{R}^N$  into Euclidean space of dimension  $N$ , in general one can not expect  $G$  to be realizable as a group of rigid motions of  $\mathbb{R}^N$  that leave  $f(M^n)$  invariant. Nevertheless, a fundamental fact for us is that in codimension  $N - n = 1$  this is indeed the case.

**Theorem 1** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n \geq 3$ , be a compact hypersurface. Then the identity component  $Iso^0(M^n)$  of the isometry group of  $M^n$  admits an orthogonal representation  $\Phi: Iso^0(M^n) \rightarrow SO(n + 1)$  such that  $f \circ g = \Phi(g) \circ f$  for all  $g \in Iso^0(M^n)$ .*

Theorem 1 may be regarded as a generalization of a classical result of Kobayashi [Ko], who proved that a compact homogeneous hypersurface of Euclidean space must

be a round sphere. In fact, the crucial step in the proof of Kobayashi's theorem is to show that the isometry group of the hypersurface can be realized as a closed subgroup of  $O(n+1)$ . The idea of the proof of Theorem 1 actually appears already in [MPST], where Euclidean  $G$ -hypersurfaces of cohomogeneity one, i.e., with principal orbits of codimension one, are considered.

We apply Theorem 1 to study compact Euclidean hypersurfaces  $f: M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n \geq 3$ , on which a connected closed subgroup  $G$  of  $Iso(M^n)$  acts locally polarly. Recall that an isometric action of a compact Lie group  $G$  on a Riemannian manifold  $M^n$  is said to be *locally polar* if the distribution of normal spaces to principal orbits on the regular part of  $M^n$  is integrable. If  $M^n$  is complete, this implies the existence of a connected complete immersed submanifold  $\Sigma$  of  $M^n$  that intersects orthogonally all  $G$ -orbits (cf. [HLO]). Such a submanifold is called a *section*, and it is always a totally geodesic submanifold of  $M^n$ . In particular, any isometric action of cohomogeneity one is locally polar. The action is said to be *polar* if there exists a closed and embedded section. Clearly, for orthogonal representations there is no distinction between polar and locally polar actions, for in this case sections are just affine subspaces.

It was shown in [BCO], Proposition 3.2.9 that if a closed subgroup of  $SO(N)$  acts polarly on  $\mathbb{R}^N$  and leaves invariant a submanifold  $f: M^n \rightarrow \mathbb{R}^N$ , then its restricted action on  $M^n$  is locally polar. Our next result states that any locally polar isometric action of a compact connected Lie group on a compact Euclidean hypersurface of dimension  $n \geq 3$  arises in this way.

**Theorem 2** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n \geq 3$ , be a compact hypersurface and let  $G$  be a closed connected subgroup of  $Iso(M^n)$  acting locally polarly on  $M^n$  with cohomogeneity  $k$ . Then there exists an orthogonal representation  $\Psi: G \rightarrow SO(n+1)$  such that  $\Psi(G)$  acts polarly on  $\mathbb{R}^{n+1}$  with cohomogeneity  $k+1$  and  $f \circ g = \Psi(g) \circ f$  for every  $g \in G$ .*

A natural problem that emerges is how to explicitly construct all compact hypersurfaces  $f: M^n \rightarrow \mathbb{R}^{n+1}$  that are invariant under a polar action of a closed subgroup  $G \subset SO(n+1)$ . This is accomplished by the following result. Recall that the Weyl group of the  $G$ -action is defined as  $W = N(\Sigma)/Z(\Sigma)$ , where  $\Sigma$  is a section,  $N(\Sigma) = \{g \in G \mid g\Sigma = \Sigma\}$  and  $Z(\Sigma) = \{g \in G \mid gp = p, \forall p \in \Sigma\}$  is the intersection of the isotropy subgroups  $G_p$ ,  $p \in \Sigma$ .

**Theorem 3** *Let  $G \subset SO(n+1)$  be a closed subgroup that acts polarly on  $\mathbb{R}^{n+1}$ , let  $\Sigma$  be a section and let  $L$  be a compact immersed hypersurface of  $\Sigma$  which is invariant under the Weyl group of the  $G$ -action. Then  $G(L)$  is a compact  $G$ -invariant immersed hypersurface of  $\mathbb{R}^{n+1}$ . Conversely, any compact hypersurface  $f: M^n \rightarrow \mathbb{R}^{n+1}$  that is invariant under a polar action of a closed subgroup of  $SO(n+1)$  can be constructed in this way.*

The simplest examples of hypersurfaces  $f: M^n \rightarrow \mathbb{R}^{n+1}$  that are invariant under a polar action on  $\mathbb{R}^{n+1}$  of a closed subgroup of  $SO(n+1)$  are the rotation hypersurfaces.

These are invariant under a polar representation which is the sum of a trivial representation on a subspace  $\mathbb{R}^k \subset \mathbb{R}^{n+1}$  and one which acts transitively on the unit sphere of its orthogonal complement  $\mathbb{R}^{n-k+1}$ . In this case any subspace  $\mathbb{R}^{k+1}$  containing  $\mathbb{R}^k$  is a section, and the Weyl group of the action is just the group  $\mathbb{Z}^2$  generated by the reflection with respect to the “axis”  $\mathbb{R}^k$ . Thus, the condition that a hypersurface  $L^k \subset \mathbb{R}^{k+1}$  be invariant under the Weyl group reduces in this case to  $L^k$  being symmetric with respect to  $\mathbb{R}^k$ . We now give several sufficient conditions for a compact Euclidean hypersurface as in Theorem 2 to be a rotation hypersurface.

**Corollary 4** *Under the assumptions of Theorem 2, any of the following additional conditions implies that  $f$  is a rotation hypersurface:*

- (i) *there exists a totally geodesic (in  $M^n$ )  $G$ -principal orbit;*
- (ii)  *$k = n - 1$ ;*
- (iii) *the  $G$ -principal orbits are umbilical in  $M^n$ ;*
- (iv) *there exists a  $G$ -principal orbit with nonzero constant sectional curvatures;*
- (v) *there exists a  $G$ -principal orbit with positive sectional curvatures.*

Moreover,  $G$  is isomorphic to one of the closed subgroups of  $SO(n - k + 1)$  that act transitively on  $\mathbb{S}^{n-k}$ .

For a list of all closed subgroups of  $SO(n)$  that act transitively on the sphere, see, e.g., [EH], p. 392. Corollary 4 generalizes similar results in [PS], [AMN] and [MPST] for compact Euclidean  $G$ -hypersurfaces of cohomogeneity one. In part (v), weakening the assumption to *non-negativity* of the sectional curvatures of some principal  $G$ -orbit implies  $f$  to be a *multi-rotational hypersurface* in the sense of [DN]:

**Corollary 5** *Under the assumptions of Theorem 2, suppose further that there exists a  $G$ -principal orbit  $Gp$  with nonnegative sectional curvatures. Then there exist an orthogonal decomposition  $\mathbb{R}^{n+1} = \bigoplus_{i=0}^k \mathbb{R}^{n_i}$  into  $\tilde{G}$ -invariant subspaces, where  $\tilde{G} = \Psi(G)$ , and connected Lie subgroups  $G_1, \dots, G_k$  of  $\tilde{G}$  such that  $G_i$  acts on  $\mathbb{R}^{n_i}$ , the action being transitive on  $\mathbb{S}^{n_i-1} \subset \mathbb{R}^{n_i}$ , and the action of  $\tilde{G} = G_1 \times \dots \times G_k$  on  $\mathbb{R}^{n+1}$  given by*

$$(g_1 \dots g_k)(v_0, v_1, \dots, v_k) = (v_0, g_1 v_1, \dots, g_k v_k)$$

*is orbit equivalent to the action of  $\tilde{G}$ . In particular, if  $Gp$  is flat then  $n_i = 2$  and  $G_i$  is isomorphic to  $SO(2)$  for  $i = 1, \dots, k$ .*

Finally, we apply some of the previous results to a problem that at first sight has no relation to isometric actions whatsoever.

Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$  be a rotation hypersurface as described in the paragraph following Theorem 3. Then the open and dense subset of  $M^n$  that is mapped by  $f$  onto the complement of the axis  $\mathbb{R}^k$  is isometric to the warped product  $L^k \times_{\rho} N^{n-k}$ , where  $N^{n-k}$  is the orbit of some fixed point  $f(p) \in L^k$  under the action of  $\tilde{G}$ , and the warping function  $\rho: L^k \rightarrow \mathbb{R}_+$  is a constant multiple of the distance to  $\mathbb{R}^k$ . Recall that a *warped product*  $N_1 \times_{\rho} N_2$  of Riemannian manifolds  $(N_1, \langle \cdot, \cdot \rangle_{N_1})$  and  $(N_2, \langle \cdot, \cdot \rangle_{N_2})$  with *warping function*  $\rho: N_1 \rightarrow \mathbb{R}_+$  is the product manifold  $N_1 \times N_2$  endowed with the metric

$$\langle \cdot, \cdot \rangle = \pi_1^* \langle \cdot, \cdot \rangle_{N_1} + (\rho \circ \pi_1)^2 \pi_2^* \langle \cdot, \cdot \rangle_{N_2},$$

where  $\pi_i: N_1 \times N_2 \rightarrow N_i$ ,  $1 \leq i \leq 2$ , denote the canonical projections.

We prove that, conversely, compact Euclidean rotation hypersurfaces of dimension  $n \geq 3$  are characterized by their warped product structure.

**Theorem 6** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n \geq 3$ , be a compact hypersurface. If there exists an isometry onto an open and dense subset  $U \subset M^n$  of a warped product  $L^k \times_{\rho} N^{n-k}$  with  $N^{n-k}$  connected and complete (in particular if  $M^n$  is isometric to a warped product  $L^k \times_{\rho} N^{n-k}$ ) then  $f$  is a rotation hypersurface.*

Theorem 6 can be seen as a global version in the hypersurface case of the local classification in [DT] of isometric immersions in codimension  $\ell \leq 2$  of warped products  $L^k \times_{\rho} N^{n-k}$ ,  $n - k \geq 2$ , into Euclidean space.

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## 2 Proof of Theorem 1

The proof of Theorem 1 relies on a result of Sacksteder [Sa] (see also the proof in [Da], Theorem 6.14, which is based on an unpublished manuscript by D. Ferus) according to which a compact hypersurface  $f: M^n \rightarrow \mathbb{R}^{n+1}$ ,  $n \geq 3$ , is rigid whenever the subset of totally geodesic points of  $f$  does not disconnect  $M^n$ . Recall that  $f$  is *rigid* if any other isometric immersion  $\tilde{f}: M^n \rightarrow \mathbb{R}^{n+1}$  differs from it by a rigid motion of  $\mathbb{R}^{n+1}$ . The proof of Sacksteder's theorem actually shows more than the preceding statement. Namely, let  $f, \tilde{f}: M^n \rightarrow \mathbb{R}^{n+1}$  be isometric immersions into  $\mathbb{R}^{n+1}$  of a compact Riemannian manifold  $M^n$ ,  $n \geq 3$ , and let  $\phi: T^{\perp} M_f^n \rightarrow T^{\perp} M_{\tilde{f}}^n$  be the vector bundle isometry between the normal bundles of  $f$  and  $\tilde{f}$  defined as follows. Given a unit vector  $\xi_x \in T_x^{\perp} M_f^n$  at  $x \in M$ , let  $\phi(\xi_x)$  be the unique unit vector in  $T_x^{\perp} M_{\tilde{f}}^n$  such that  $\{f_* X_1, \dots, f_* X_n, \xi_x\}$  and  $\{\tilde{f}_* X_1, \dots, \tilde{f}_* X_n, \phi(\xi_x)\}$  determine the same orientation in  $\mathbb{R}^{n+1}$ , where  $\{X_1, \dots, X_n\}$  is any ordered basis of  $T_x M^n$ . Then it is shown that

$$\alpha_{\tilde{f}}(x) = \pm \phi(x) \circ \alpha_f(x) \tag{1}$$

at each point  $x \in M^n$ , where  $\alpha_f$  and  $\alpha_{\tilde{f}}$  denote the second fundamental forms of  $f$  and  $\tilde{f}$ , respectively, with values in the normal bundle. The proof is based on a careful analysis of the distributions on  $M^n$  determined by the *relative nullity subspaces*  $\Delta(x) = \ker \alpha_f(x)$  and  $\tilde{\Delta}(x) = \ker \alpha_{\tilde{f}}(x)$  of  $f$  and  $\tilde{f}$ , respectively. Rigidity of  $f$  under the assumption that the subset of totally geodesic points of  $f$  does not disconnect  $M^n$  then follows immediately from the Fundamental Theorem of Hypersurfaces (cf. [Da], Theorem 1.1).

*Proof of Theorem 1.* Given  $g \in Iso^0(M^n)$ , let  $\alpha_{f \circ g}$  denote the second fundamental form of  $f \circ g$ . We claim that

$$\alpha_{f \circ g}(x) = \phi_g(x) \circ \alpha_f(x) \quad (2)$$

for every  $g \in Iso^0(M^n)$  and  $x \in M^n$ , where  $\phi_g$  denotes the vector bundle isometry between  $T^\perp M_f^n$  and  $T^\perp M_{f \circ g}^n$  defined as in the preceding paragraph. On one hand,

$$\alpha_{f \circ g}(x)(X, Y) = \alpha_f(gx)(g_*X, g_*Y) \quad (3)$$

for every  $g \in Iso^0(M^n)$ ,  $x \in M^n$  and  $X, Y \in T_x M^n$ . In particular, this implies that for any fixed  $x \in M^n$  the map  $\Theta_x: Iso^0(M^n) \rightarrow \text{Sym}(T_x M^n \times T_x M^n \rightarrow T_x^\perp M_f^n)$  into the vector space of symmetric bilinear maps of  $T_x M^n \times T_x M^n$  into  $T_x^\perp M_f^n$ , given by

$$\Theta_x(g)(X, Y) = \phi_g(x)^{-1}(\alpha_{f \circ g}(x)(X, Y)) = \phi_g(x)^{-1}(\alpha_f(gx)(g_*X, g_*Y))$$

for any  $X, Y \in T_x M^n$ , is continuous. On the other hand, by the preceding remarks on Sacksteder's theorem, either  $\alpha_{f \circ g}(x) = \phi_g(x) \circ \alpha_f(x)$  or  $\alpha_{f \circ g}(x) = -\phi_g(x) \circ \alpha_f(x)$ . Thus  $\Theta_x$  is a continuous map taking values in  $\{\alpha_f(x), -\alpha_f(x)\}$ , hence it must be constant because  $Iso^0(M^n)$  is connected. Since  $\Theta_x(\text{id}) = \alpha_f(x)$ , our claim follows.

We conclude that for each  $g \in Iso^0(M^n)$  there exists a rigid motion  $\tilde{g} \in Iso(\mathbb{R}^{n+1})$  such that  $f \circ g = \tilde{g} \circ f$ . It now follows from standard arguments that  $g \mapsto \tilde{g}$  defines a Lie-group homomorphism  $\Phi: Iso^0(M^n) \rightarrow Iso(\mathbb{R}^{n+1})$ , whose image must lie in  $SO(n+1)$  because it is compact and connected. ■

**Remarks 7** (i) Theorem 1 is also true for compact hypersurfaces of dimension  $n \geq 3$  of hyperbolic space, as well as for *complete* hypersurfaces of dimension  $n \geq 4$  of the sphere. It also holds for complete hypersurfaces of dimension  $n \geq 3$  of both Euclidean and hyperbolic spaces, under the additional assumption that they do not carry a complete leaf of dimension  $(n-1)$  or  $(n-2)$  of their relative nullity distributions. In fact, the proof of Theorem 1 carries over in exactly the same way for these cases, because so does equation (1) (cf. [Da], p. 96-100).

(ii) Clearly, Theorem 1 does not hold for isometric immersions  $f: M^n \rightarrow \mathbb{R}^{n+\ell}$  of codimension  $\ell \geq 2$ . Namely, counterexamples can be easily constructed, for instance, by means of compositions  $f = h \circ g$  of isometric immersions  $g: M^n \rightarrow \mathbb{R}^{n+1}$  and  $h: V \rightarrow \mathbb{R}^{n+\ell}$ , with  $V \subset \mathbb{R}^{n+1}$  an open subset containing  $g(M^n)$ .

The next result gives a sufficient condition for rigidity of a compact hypersurface  $f: M^n \rightarrow \mathbb{R}^{n+1}$  as in Theorem 1 in terms of  $\tilde{G} = \Phi(Iso^0(M^n))$ .

**Proposition 8** *Under the assumptions of Theorem 1, suppose that  $\tilde{G} = \Phi(Iso^0(M^n))$  does not have a fixed vector. Then  $f$  is free of totally geodesic points. In particular,  $f$  is rigid.*

*Proof:* Suppose that the subset  $B$  of totally geodesic points of  $f$  is nonempty. Since  $\alpha_{f \circ g}(x) = \phi_g(x) \circ \alpha_f(x)$  for every  $g \in G = Iso^0(M^n)$  and  $x \in M^n$  by (2),  $B$  coincides with the set of totally geodesic points of  $f \circ g$  for every  $g \in G$ . In view of (3), this amounts to saying that  $B$  is  $G$ -invariant. Thus, if  $Gp$  is the orbit of a point  $p \in B$  then  $Gp \subset B$ . Since  $Gp$  is connected, it follows from [DG], Lemma 3.14 that  $f(Gp)$  is contained in a hyperplane  $\mathcal{H}$  that is tangent to  $f$  along  $Gp$ . Therefore, a unit vector  $v$  orthogonal to  $\mathcal{H}$  spans  $T_{gp}^\perp M_f^n$  for every  $gp \in Gp$ . Since  $\tilde{g}T_p^\perp M_f^n = T_{gp}^\perp M_f^n$  for every  $\tilde{g} = \Phi(g) \in \tilde{G}$ , because  $f$  is equivariant with respect to  $\Phi$ , the connectedness of  $\tilde{G}$  implies that it must fix  $v$ . ■

Proposition 8 implies, for instance, that if a closed connected subgroup of  $Iso(M^n)$  acts on  $M^n$  with cohomogeneity one then either  $f$  is a rotation hypersurface over a plane curve or it is free of totally geodesic points, and in particular it is rigid (see [MPST], Theorem 1). As another consequence we have:

**Corollary 9** *Let  $f: M^3 \rightarrow \mathbb{R}^4$  be a compact hypersurface. If  $f$  has a totally geodesic point (in particular, if it is not rigid) then either  $M^3$  has finite isometry group or  $f$  is a rotation hypersurface.*

*Proof:* Let  $\Phi: Iso^0(M^3) \rightarrow SO(4)$  be the orthogonal representation given by Theorem 1. By Proposition 8, if  $f$  has a totally geodesic point then  $\tilde{G} = \Phi(Iso^0(M^3))$  has a fixed vector  $v$ , hence it can be regarded as a subgroup of  $SO(3)$ . Therefore, either the restricted action of  $\tilde{G}$  on  $\{v\}^\perp$  has also a fixed vector or it is transitive on the sphere. In the first case, either  $Iso^0(M^3)$  is trivial, that is,  $Iso(M^3)$  is finite, or  $\tilde{G}$  fixes a two dimensional subspace  $\mathbb{R}^2$  of  $\mathbb{R}^4$ , in which case  $f$  is a rotation hypersurface over a surface in a half-space  $\mathbb{R}_+^3$  with  $\mathbb{R}^2$  as boundary. In the latter case,  $f$  is a rotation hypersurface over a plane curve in a half-space  $\mathbb{R}_+^2$  having  $\text{span}\{v\}$  as boundary. ■

### 3 Proof of Theorem 2

For the proof of Theorem 2, we recall from Theorem 1 that there exists an orthogonal representation  $\Phi: Iso^0(M^n) \rightarrow SO(n+1)$  such that  $f \circ g = \Phi(g) \circ f$  for every  $g \in Iso^0(M^n)$ . Since  $G$  is connected we have  $G \subset Iso^0(M^n)$ , hence it suffices to prove that  $\tilde{G} = \Phi(G)$  acts polarly on  $\mathbb{R}^{n+1}$  with cohomogeneity  $k+1$  and then set  $\Psi = \Phi|_G$ .

We claim that there exists a principal orbit  $Gp$  such that the position vector  $f$  is nowhere tangent to  $f(M^n)$  along  $Gp$ , that is,  $f(g(p)) \notin f_*(g(p))T_{g(p)}M^n$  for any  $g \in G$ . In order to prove our claim we need the following observation.

**Lemma 10** *Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$  be a hypersurface. Assume that the position vector is tangent to  $f(M^n)$  on an open subset  $U \subset M^n$ . Then the index of relative nullity  $\nu_f(x) = \dim \Delta_f(x)$  of  $f$  is positive at any point  $x \in U$ .*

*Proof:* Let  $Z$  be a vector field on  $U$  such that  $f_*(p)Z(p) = f(p)$  for any  $p \in U$  and let  $\eta$  be a local unit normal vector field to  $f$ . Differentiating  $\langle \eta, f \rangle = 0$  yields  $\langle AX, Z \rangle = 0$  for any tangent vector field  $X$  on  $U$ , where  $A$  denotes the shape operator of  $f$  with respect to  $\eta$ . Thus  $AZ = 0$ . ■

Going back to the proof of the claim, since  $M^n$  is a compact Riemannian manifold isometrically immersed in Euclidean space as a hypersurface, there exists an open subset  $V \subset M^n$  where the sectional curvatures of  $M^n$  are strictly positive. In particular, the index of relative nullity of  $f$  vanishes at every  $x \in V$ . If the position vector were tangent to  $f(M^n)$  at every regular point of  $V$ , it would be tangent to  $f(M^n)$  everywhere on  $V$ , because the set of regular points is dense on  $M^n$ . This is in contradiction with Lemma 10 and shows that there must exist a regular point  $p \in V$  such that  $f(p) \notin f_*(p)T_pM^n$ . Since  $f$  is equivariant, we must have in fact that  $f(g(p)) \notin f_*(g(p))T_{g(p)}M^n$  for any  $g \in G$  and the claim is proved.

Now let  $Gp$  be a principal orbit such that the position vector  $f$  is nowhere tangent to  $f(M^n)$  along  $Gp$ . Then the normal bundle of  $f|_{Gp}$  splits (possibly non-orthogonally) as  $\text{span}\{f\} \oplus f_*T^\perp Gp$ . Let  $\xi$  be an equivariant normal vector field to  $Gp$  in  $M^n$  and let  $\eta = f_*\xi$ . Then, denoting  $\Phi(g)$  by  $\tilde{g}$  and identifying  $\tilde{g}$  with its derivative at any point, because it is linear, we have

$$\tilde{g}\eta(p) = (f \circ g)_*(p)\xi(p) = f_*(gp)g_*(p)\xi(p) = f_*(gp)\xi(gp) = \eta(gp) \quad (4)$$

for any  $g \in G$ . In particular,  $\langle \eta(gp), f(gp) \rangle = \langle \tilde{g}\eta(p), \tilde{g}f(p) \rangle = \langle \eta(p), f(p) \rangle$  for every  $g \in G$ , that is,  $\langle \eta, f \rangle$  is constant on  $Gp$ . It follows that

$$X\langle \eta, f \rangle = 0, \text{ for any } X \in TGp, \quad (5)$$

and hence

$$\langle \tilde{\nabla}_X \eta, f \rangle = X\langle \eta, f \rangle - \langle \xi, X \rangle = 0, \quad (6)$$

where  $\tilde{\nabla}$  denotes the derivative in  $\mathbb{R}^{n+1}$ . On the other hand, since  $G$  acts locally polarly on  $M^n$ , we have that  $\xi$  is parallel in the normal connection of  $Gp$  in  $M$ . Therefore

$$(\tilde{\nabla}_X \eta)_{f_*T^\perp Gp} = f_*(\nabla_X \xi)_{T^\perp Gp} = 0, \quad (7)$$

where  $\nabla$  is the Levi-Civita connection of  $M^n$ ; here, writing a vector subbundle as a subscript of a vector field indicates taking its orthogonal projection onto that subbundle.

It follows from (6) and (7) that  $\eta$  is parallel in the normal connection of  $f|_{Gp}$ . On the other hand, the position vector  $f$  is clearly also an equivariant normal vector field of  $f|_{Gp}$  which is parallel in the normal connection.

Thus, we have shown that there exist equivariant normal vector fields to  $\tilde{G}(f(p)) = f(Gp)$  that form a basis of the normal spaces at each point and which are parallel in the normal connection. The statement now follows from the next known result, a proof of which is included for completeness.

**Lemma 11** *Let  $\tilde{G} \subset SO(n+1)$  have an orbit  $\tilde{G}(q)$  along which there exist equivariant normal vector fields that form a basis of the normal spaces at each point and which are parallel in the normal connection. Then  $\tilde{G}$  acts polarly.*

*Proof:* Since there exist equivariant normal vector fields to  $\tilde{G}(q)$  that form a basis of the normal spaces at each point, the isotropy group acts trivially on each normal space, hence  $\tilde{G}(q)$  is a principal orbit. We now show that the normal space  $T_q^\perp \tilde{G}(q)$  is a section, for which it suffices to show that any Killing vector field  $X$  induced by  $\tilde{G}$  is everywhere orthogonal to  $T_q^\perp \tilde{G}(q)$ . Given  $\xi_q \in T_q^\perp \tilde{G}(q)$ , let  $\xi$  be an equivariant normal vector field to  $\tilde{G}(q)$  extending  $\xi_q$ , which is also parallel in the normal connection. Then, denoting by  $\phi_t$  the flow of  $X$  and setting  $c(t) = \phi_t(q)$  we have

$$\left( \frac{d}{dt} \Big|_{t=0} \phi_t(\xi_q) \right)_{T_q^\perp \tilde{G}(q)} = \left( \frac{D}{dt} \Big|_{t=0} \xi(c(t)) \right)_{T_q^\perp \tilde{G}(q)} = \nabla_{X(q)}^\perp \xi = 0,$$

where  $\frac{D}{dt}$  denotes the covariant derivative in Euclidean space along  $c(t)$ . ■

**Remark 12** A closed subgroup  $G \subset SO(n+\ell)$ ,  $\ell \geq 2$ , that acts non-polarly on  $\mathbb{R}^{n+\ell}$  may leave invariant a compact submanifold  $f: M^n \rightarrow \mathbb{R}^{n+\ell}$  and induce a locally polar action on  $M^n$ . For instance, consider a compact submanifold  $f: M^n \rightarrow \mathbb{R}^{n+2}$  that is invariant by the action of a closed subgroup  $G \subset SO(n+2)$  that acts non-polarly on  $\mathbb{R}^{n+2}$  with cohomogeneity three. Then the induced action of  $G$  on  $M^n$  has cohomogeneity one, whence is locally polar. Moreover, taking  $f$  as a compact hypersurface of  $\mathbb{S}^{n+1}$  shows also that Theorem 2 is no longer true if  $\mathbb{R}^{n+1}$  is replaced by  $\mathbb{S}^{n+1}$ .

Theorem 2 yields the following obstruction for the existence of an isometric immersion in codimension one into Euclidean space of a compact Riemannian manifold acted on locally polarly by a closed connected Lie subgroup of its isometry group.

**Corollary 13** *Let  $M^n$  be a compact Riemannian manifold of dimension  $n \geq 3$  acted on locally polarly by a closed connected subgroup  $G$  of its isometry group. If  $G$  has an exceptional orbit then  $M^n$  can not be isometrically immersed in Euclidean space as a hypersurface.*



*Proof:* Let  $f: M^n \rightarrow \mathbb{R}^{n+1}$  be an isometric immersion of a compact Riemannian manifold acted on locally polarly by a closed connected subgroup  $G$  of its isometry group. We will prove that  $G$  can not have any exceptional orbit. By Theorem 2 there exists an orthogonal representation  $\Psi: G \rightarrow SO(n+1)$  such that  $\tilde{G} = \Psi(G)$  acts polarly on  $\mathbb{R}^{n+1}$  with cohomogeneity  $k+1$  and  $f \circ g = \Psi(g) \circ f$  for every  $g \in G$ . Let  $Gp$  be a nonsingular orbit. Then  $Gp$  has maximal dimension among all  $G$ -orbits, and hence  $\tilde{G}f(p) = f(Gp)$  has maximal dimension among all  $\tilde{G}$ -orbits. Since polar representations are known to admit no exceptional orbits (cf. [BCO], Corollary 5.4.3), it follows that  $\tilde{G}f(p)$  is a principal orbit. Then, for any  $g$  in the isotropy subgroup  $G_p$  we have that  $\tilde{g} = \Psi(g) \in \tilde{G}_{f(p)}$ , thus for any  $\xi_p \in T_p^\perp Gp$  we obtain

$$f_*(p)\xi_p = \tilde{g}_*f_*(p)\xi_p = (\tilde{g} \circ f)_*(p)\xi_p = (f \circ g)_*(p)\xi_p = f_*(gp)g_*\xi_p = f_*(p)g_*\xi_p.$$

Since  $f_*(p)$  is injective, then  $g_*\xi_p = \xi_p$ . This shows that the slice representation, that is, the action of the isotropy group  $G_p$  on the normal space  $T_p^\perp G(p)$  to the orbit  $G(p)$  at  $p$ , is trivial. Thus  $Gp$  is a principal orbit. ■

## 4 Isoparametric submanifolds

We now recall some results on isoparametric submanifolds and derive a few additional facts on them that will be needed for the proofs of Theorem 3 and Corollaries 4 and 5.

Given an isometric immersion  $f: M^n \rightarrow \mathbb{R}^N$  with flat normal bundle, it is well-known (cf. [St]) that for each point  $x \in M^n$  there exist an integer  $s = s(x) \in \{1, \dots, n\}$  and a uniquely determined subset  $H_x = \{\eta_1, \dots, \eta_s\}$  of  $T_x^\perp M_f^n$  such that  $T_x M^n$  is the orthogonal sum of the nontrivial subspaces

$$E_{\eta_i}(x) = \{X \in T_x M^n : \alpha(X, Y) = \langle X, Y \rangle \eta_i, \text{ for all } Y \in T_x M^n\}, \quad 1 \leq i \leq s.$$

Therefore, the second fundamental form of  $f$  has the simple representation

$$\alpha(X, Y) = \sum_{i=1}^s \langle X_i, Y_i \rangle \eta_i, \quad (8)$$

or equivalently,

$$A_\xi X = \sum_{i=1}^s \langle \xi, \eta_i \rangle X_i, \quad (9)$$

where  $X \mapsto X_i$  denotes orthogonal projection onto  $E_{\eta_i}$ . Each  $\eta_i \in H_x$  is called a *principal normal* of  $f$  at  $x$ . The Gauss equation takes the form

$$R(X, Y) = \sum_{i,j=1}^s \langle \eta_i, \eta_j \rangle X_i \wedge Y_j, \quad (10)$$

where  $(X_i \wedge Y_j)Z = \langle Z, Y_j \rangle X_i - \langle Z, X_i \rangle Y_j$ .

**Lemma 14** *Let  $f: M^n \rightarrow \mathbb{R}^N$  be an isometric immersion with flat normal bundle of a Riemannian manifold with constant sectional curvature  $c$ . Let  $\eta_1, \dots, \eta_s$  be the distinct principal normals of  $f$  at  $x \in M^n$ . Then*

- (i) *There exists at most one  $\eta_i$  such that  $|\eta_i|^2 = c$ .*
- (ii) *For all  $i, j, k \in \{1, \dots, s\}$  with  $i \neq j \neq k \neq i$  the vectors  $\eta_i - \eta_k$  and  $\eta_j - \eta_k$  are linearly independent.*

*Proof:* It follows from (10) that  $\langle \eta_i, \eta_j \rangle = c$  for all  $i, j \in \{1, \dots, s\}$  with  $i \neq j$ . If  $|\eta_i|^2 = |\eta_j|^2 = c$ , this gives  $|\eta_i - \eta_j|^2 = 0$ .

(iii) Assume that there exist  $\lambda \neq 0$  and  $i, j, k \in \{1, \dots, s\}$  with  $i \neq j \neq k \neq i$  such that  $\eta_i - \eta_k = \lambda(\eta_j - \eta_k)$ . Then

$$|\eta_i|^2 - c = \langle \eta_i - \eta_k, \eta_i \rangle = \lambda \langle \eta_j - \eta_k, \eta_i \rangle = 0,$$

and similarly  $|\eta_j|^2 = c$ , in contradiction with (i). ■

Let  $f: M^n \rightarrow \mathbb{R}^N$  be an isometric immersion with flat normal bundle and let  $H_x = \{\eta_1, \dots, \eta_s\}$  be the set of principal normals of  $f$  at  $x \in M^n$ . If the map  $M^n \rightarrow \{1, \dots, n\}$  given by  $x \mapsto \#H_x$  has a constant value  $s$  on an open subset  $U \subset M^n$ , there exist smooth normal vector fields  $\eta_1, \dots, \eta_s$  on  $U$  such that  $H_x = \{\eta_1(x), \dots, \eta_s(x)\}$  for any  $x \in U$ . Furthermore, each  $E_{\eta_i} = (E_{\eta_i}(x))_{x \in U}$  is a  $C^\infty$ -subbundle of  $TU$  for  $1 \leq i \leq s$ . The following result is contained in [DN], Lemma 2.3.

**Lemma 15.** *Let  $f: M^n \rightarrow \mathbb{R}^{n+p}$  be an isometric immersion with flat normal bundle and a constant number  $s$  of principal normals  $\eta_1, \dots, \eta_s$  everywhere. Assume that for a fixed  $i \in \{1, \dots, s\}$  all principal normals  $\eta_j$ ,  $j \neq i$ , are parallel in the normal connection along  $E_{\eta_i}$  and that the vectors  $\eta_i - \eta_j$  and  $\eta_i - \eta_\ell$  are everywhere pairwise linearly independent for any pair of indices  $1 \leq j \neq \ell \leq s$  with  $j, \ell \neq i$ . Then  $E_{\eta_i}^\perp$  is totally geodesic.*

*Proof:* The Codazzi equation yields

$$\langle \nabla_{X_\ell} X_j, X_i \rangle (\eta_i - \eta_j) = \langle \nabla_{X_\ell} X_j, X_i \rangle (\eta_i - \eta_\ell), \quad i \neq j \neq k \neq i, \quad (11)$$

and

$$\nabla_{X_i}^\perp \eta_j = \langle \nabla_{X_j} X_j, X_i \rangle (\eta_j - \eta_i), \quad i \neq j, \quad (12)$$

for all unit vectors  $X_i \in E_{\eta_i}$ ,  $X_j \in E_{\eta_j}$  and  $X_\ell \in E_{\eta_\ell}$ . ■

An isometric immersion  $f: M^n \rightarrow \mathbb{R}^N$  is called *isoparametric* if it has flat normal bundle and the principal curvatures of  $f$  with respect to every parallel normal vector field along any curve in  $M^n$  are constant (with constant multiplicities). The following facts on isoparametric submanifolds are due to Strübing [St].

**Theorem 16** *Let  $f: M^n \rightarrow \mathbb{R}^N$  be an isoparametric isometric immersion. Then*

- (i) *The number of principal normals is constant on  $M^n$ .*
- (ii) *The first normal spaces, i.e., the subspaces of the normal spaces spanned by the image of the second fundamental form, determine a parallel subbundle of the normal bundle.*
- (iii) *The subbundles  $E_{\eta_i}$ ,  $1 \leq i \leq s$ , are totally geodesic and the principal normals  $\eta_1, \dots, \eta_s$  are parallel in the normal connection.*
- (iv) *The subbundles  $E_{\eta_i}$ ,  $1 \leq i \leq s$ , are parallel if and only if  $f$  has parallel second fundamental form.*
- (v) *If the principal normals  $\eta_1, \dots, \eta_s$  satisfy  $\langle \eta_i, \eta_j \rangle \geq 0$  everywhere then  $f$  has parallel second fundamental form and  $\langle \eta_i, \eta_j \rangle = 0$  everywhere.*

The next result will be used in the proofs of Corollaries 4 and 5.

**Proposition 17** *Let  $f: M^n \rightarrow \mathbb{R}^N$ ,  $n \geq 2$ , be a compact isoparametric submanifold.*

- (i) *If  $M^n$  has constant sectional curvature  $c$ , then either  $c > 0$  and  $f(M^n)$  is a round sphere or  $c = 0$  and  $f(M^n)$  is an extrinsic product of circles.*
- (ii) *If  $M^n$  has nonnegative sectional curvatures, then  $f(M^n)$  is an extrinsic product of round spheres or circles. In particular, if  $M^n$  has positive sectional curvatures then  $f(M^n)$  is a round sphere.*

*Proof:* By Theorem 16, the number of distinct principal normals  $\eta_1, \dots, \eta_s$  of  $f$  is constant on  $M^n$  and all of them are parallel in the normal connection. Moreover, the subbundles  $E_{\eta_i}$ ,  $1 \leq i \leq s$ , are totally geodesic. If  $M^n$  has constant sectional curvature  $c$ , then it follows from Lemmas 14 and 15 that also  $E_{\eta_i}^\perp$  is totally geodesic for  $1 \leq i \leq s$ , and hence the sectional curvatures along planes spanned by vectors in different subbundles vanish. Therefore  $c = 0$ , unless  $s = 1$  and  $f$  is umbilic, in which case  $c > 0$  and  $f(M^n)$  is a round sphere. Furthermore, if  $c = 0$  and  $E_{\eta_i}$  has rank at least 2 for some  $1 \leq i \leq s$  then the sectional curvature along a plane tangent to  $E_{\eta_i}$  is  $|\eta_i|^2 = 0$ , in contradiction with the compactness of  $M^n$ . Hence  $E_{\eta_i}$  has rank 1 for  $1 \leq i \leq s$ . We conclude that the universal covering of  $M^n$  is isometric to  $\mathbb{R}^n$ , and that  $f \circ \pi$  splits as a product of circles by Moore's Lemma [Mo], where  $\pi: \mathbb{R}^n \rightarrow M^n$  is the covering map.

Assume now that  $M^n$  has nonnegative sectional curvatures. It follows from (10) that  $\langle \eta_i, \eta_j \rangle \geq 0$  for  $1 \leq i \neq j \leq s$ , whence  $f$  has parallel second fundamental form and all subbundles  $E_{\eta_i}$ ,  $1 \leq i \leq s$ , are parallel by parts (iv) and (v) of Theorem 16. We obtain from the de Rham decomposition theorem that the universal covering of  $M^n$  splits isometrically as  $M_1^{n_1} \times \dots \times M_s^{n_s}$ , where each factor  $M_i^{n_i}$  is either  $\mathbb{R}$  if  $n_i = 1$  or a

sphere  $\mathbb{S}_i^{n_i}$  of curvature  $|\eta_i|^2$  if  $n_i \geq 2$ . Moreover, if  $\pi: M_1^{n_1} \times \cdots \times M_s^{n_s} \rightarrow M^n$  denotes the covering map, then Moore's Lemma implies that  $f \circ \pi$  splits as  $f \circ \pi = f_1 \times \cdots \times f_s$ , where  $f_i(M_i^{n_i})$  is a round sphere or circle for  $1 \leq i \leq s$ . ■

To every compact isoparametric submanifold  $f: M^n \rightarrow \mathbb{R}^N$  one can associate a finite group, its Weyl group, as follows. Let  $\eta_1, \dots, \eta_g$  denote the principal normal vector fields of  $f$ . For  $p \in M^n$ , let  $H_j(p)$ ,  $1 \leq j \leq g$ , be the focal hyperplane of  $T_p^\perp M^n$  given by the equation  $\langle \eta_j(p), \cdot \rangle = 1$ . Then one can show that the reflection on the affine normal space  $p + T_p^\perp M^n$  with respect to each affine focal hyperplane  $p + H_i(p)$  leaves  $\bigcup_{j=1}^g (p + H_j(p))$  invariant, and thus the set of all such reflections generate a finite group, the Weyl group of  $f$  at  $p$ . Moreover, the Weyl groups of  $f$  at different points are conjugate by the parallel transport with respect to the normal connection, hence a well-defined Weyl group  $W$  can be associated to  $f$ . We refer to [PT<sub>2</sub>] for details. In the proof of Theorem 3 we will need the following property of the Weyl group of an isoparametric submanifold.

**Proposition 18** *Let  $f: M^n \rightarrow \mathbb{R}^N$  be a compact isoparametric submanifold and let  $W(p)$  be its Weyl group at  $p \in M^n$ . Assume that  $W(p)$  leaves invariant an affine hyperplane  $\mathcal{H}$  orthogonal to  $\xi \in T_p^\perp M$ . Then  $f(M^n)$  is contained in the affine hyperplane of  $\mathbb{R}^N$  through  $p$  orthogonal to  $\xi$ .*

*Proof:* It follows from the assumption that  $\mathcal{H}$  is orthogonal to every focal hyperplane  $p + H_j(p)$ ,  $1 \leq j \leq g$ , of  $f$  at  $p$ . For  $q \in \mathcal{H}$ , let  $Q = \sum_{g \in W(p)} gq \in \mathcal{H}$ . Then  $Q$  is a fixed point of  $W(p)$ , hence it lies in the intersection  $\bigcap_{j=1}^g (p + H_j(p))$  of all affine focal hyperplanes of  $f$  at  $p$ . We obtain that the line through  $Q$  orthogonal to  $\mathcal{H}$  lies in  $\bigcap_{j=1}^g (p + H_j(p))$ . Therefore  $\langle \eta_j, Q + \lambda \xi \rangle = 1$  for every  $\lambda \in \mathbb{R}$ ,  $1 \leq j \leq g$ , which implies that  $\langle \eta_j, \xi \rangle = 0$ , for every  $1 \leq j \leq g$ . Now extend  $\xi$  to a parallel vector field along  $M^n$  with respect to the normal connection. Since the principal normal vector fields  $\eta_1, \dots, \eta_g$  of  $f$  are parallel with respect to the normal connection by Theorem 16-(iii), it follows that  $\langle \eta_j, \xi \rangle = 0$  everywhere, and hence the shape operator  $A_\xi$  of  $f$  with respect to  $\xi$  is identically zero by (9). Then  $\xi$  is constant in  $\mathbb{R}^N$  and the conclusion follows. ■

A rich source of isoparametric submanifolds is provided by the following result of Palais and Terng (see [PT<sub>1</sub>], Theorem 6.5).

**Proposition 19** *If a closed subgroup  $G \subset SO(N)$  acts polarly on  $\mathbb{R}^N$  then any of its principal orbits is an isoparametric submanifold of  $\mathbb{R}^N$ .*

To conclude this section, we point out that if  $G \subset SO(N)$  acts polarly on  $\mathbb{R}^N$  and  $\Sigma$  is a section, then the Weyl group  $W = N(\Sigma)/Z(\Sigma)$  of the  $G$ -action coincides with the Weyl group  $W(p)$  just defined of any principal orbit  $Gp$ ,  $p \in \Sigma$ , as an isoparametric submanifold of  $\mathbb{R}^N$  (cf. [PT<sub>2</sub>]).

## 5 Proof of Theorem 3

We first prove the converse. Let  $G \subset SO(n+1)$  act polarly on  $\mathbb{R}^{n+1}$ , let  $\Sigma$  be a section of the  $G$ -action and let  $M^n \subset \mathbb{R}^{n+1}$  be a  $G$ -invariant immersed hypersurface. It suffices to prove that  $\Sigma$  is transversal to  $M^n$ , for then  $L = \Sigma \cap M^n$  is a compact hypersurface of  $\Sigma$  that is invariant under the Weyl group  $W$  of the action and  $M^n = G(L)$ .

Assume, on the contrary, that transversality does not hold. Then there exists  $p \in \Sigma \cap M^n$  with  $T_p \Sigma \subset T_p M^n$ . Fix  $v \in T_p \Sigma$  in a principal orbit of the slice representation at  $p$  and let  $\gamma: (-\epsilon, \epsilon) \rightarrow M^n$  be a smooth curve with  $\gamma(0) = p$  and  $\gamma'(0) = v$ . Since  $M^n$  is  $G$ -invariant, it contains  $\{g\gamma(t) \mid g \in G_p, t \in (-\epsilon, \epsilon)\}$ . Therefore,  $T_p M^n$  contains  $G_p v$ , and hence  $\mathbb{R}v \oplus T_v G_p v$ . Recall that  $T_p \Sigma$  is a section for the slice representation at  $p$  (see [PT<sub>1</sub>], Theorem 4.6). Therefore  $T_v G_p v$  is a subspace of  $T_p^\perp G(p)$  orthogonal to  $T_p \Sigma$  with

$$\dim T_v G_p v = \dim T_p^\perp G(p) - \dim \Sigma. \quad (13)$$

Moreover, again by the  $G$ -invariance of  $M^n$ , we have that  $G(p) \subset M^n$ , and hence  $T_p G(p) \subset T_p M^n$ . Using (13), we conclude that

$$\begin{aligned} \dim T_p M^n &\geq \dim G(p) + \dim \Sigma + \dim T_v G_p v \\ &= \dim G(p) + \dim T_p^\perp G(p) = n + 1, \end{aligned}$$

a contradiction.

In order to prove the direct statement, it suffices to show that at each point  $p \in L$  which is a singular point of the  $G$ -action the subset

$$H = \{\gamma'(0) \mid \gamma: (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{n+1} \text{ is a curve in } \mathbb{R}^{n+1} \text{ with } \gamma(-\epsilon, \epsilon) \subset M^n \text{ and } \gamma(0) = p\}$$

is an  $n$ -dimensional subspace of  $\mathbb{R}^{n+1}$ , the tangent space of  $M^n$  at  $p$ . Clearly, we have

$$H = T_p G(p) \bigcup_{g \in G_p} g T_p L.$$

We use again that  $T_p \Sigma$  is a section of the slice representation at  $p$  and that, in addition, the Weyl group for the slice representation is  $W(\Sigma)_p = W \cap G_p$  ([PT<sub>1</sub>], Theorem 4.6). Since  $L$  is invariant under  $W$  by assumption, it follows that  $T_p L$  is invariant under  $W(\Sigma)_p$ . Let  $\xi \in \Sigma$  be a unit vector normal to  $L$  at  $p$ . Then, for any principal vector  $v \in T_p L \subset T_p \Sigma$  of the slice representation at  $p$  it follows from Proposition 18 that  $G_p v$  lies in the affine hyperplane  $\pi$  of  $\mathbb{R}^N$  through  $p$  orthogonal to  $\xi$ . Therefore  $g T_p L \subset \pi$  for every  $g \in G_p$ . Since  $T_p G(p)$  is orthogonal to  $\Sigma$ , we conclude that  $H \subset \pi$ , and hence  $H = \pi$  by dimension reasons. ■

**Remark 20** Theorems 1 and 3 yield as a special case the main theorem in [MPST]: any compact hypersurface of  $\mathbb{R}^{n+1}$  with cohomogeneity one under the action of a closed

connected subgroup of its isometry group is given as  $G(\gamma)$ , where  $G \subset SO(n+1)$  acts on  $\mathbb{R}^{n+1}$  with cohomogeneity two (hence polarly) and  $\gamma$  is a smooth curve in a (two-dimensional) section  $\Sigma$  which is invariant under the Weyl group  $W$  of the  $G$ -action. We take the opportunity to point out that starting with a smooth curve  $\beta$  in a Weyl chamber  $\sigma$  of  $\Sigma$  (which is identified with the orbit space of the  $G$ -action) which is orthogonal to the boundary  $\partial\sigma$  of  $\sigma$  is not enough to ensure smoothness of  $\gamma = W(\beta)$ , or equivalently, of  $G(\beta)$ , as claimed in [MPST]. One should require, in addition, that after expressing  $\beta$  locally as a graph  $z = f(x)$  over its tangent line at a point of intersection with  $\partial\sigma$ , the function  $f$  be *even*, that is, all of its derivatives of odd order vanish and not only the first one.

## 6 Proofs of Corollaries 4 and 5.

For the proof of Corollary 4 we need another known property of polar representations, a simple proof of which is included for the sake of completeness

**Lemma 21** *Let  $\tilde{G} \subset SO(n+1)$  act polarly and have a principal orbit  $\tilde{G}(q)$  that is not full, i.e., the linear span  $V$  of  $\tilde{G}(q)$  is a proper subspace of  $\mathbb{R}^{n+1}$ . Then  $\tilde{G}$  acts trivially on  $V^\perp$ .*

*Proof:* Let  $v \in V^\perp$ . Then  $v$  belongs to the normal spaces of  $\tilde{G}(q)$  at any point, hence admits a unique extension to an equivariant normal vector field  $\xi$  along  $\tilde{G}(q)$ . Moreover, the shape operator  $A_\xi$  of  $\tilde{G}(q)$  with respect to  $\xi$  is identically zero, for  $A_v$  is clearly zero and  $\xi$  is equivariant. Now, since  $\tilde{G}$  acts polarly, the vector field  $\xi$  is parallel in the normal connection. It follows that  $\xi$  is a constant vector in  $\mathbb{R}^{n+1}$ , which means that  $\tilde{G}$  fixes  $v$ . ■

*Proof of Corollary 4.* We know from Theorem 2 that there exists an orthogonal representation  $\Psi: G \rightarrow SO(n+1)$  such that  $\tilde{G} = \Psi(G)$  acts polarly on  $\mathbb{R}^{n+1}$  and  $f \circ g = \Psi(g) \circ f$  for every  $g \in G$ . We claim that any of the conditions in the statement implies that  $f$  immerses some  $G$ -principal orbit  $Gp$  in  $\mathbb{R}^{n+1}$  as a round sphere (circle, if  $k = n-1$ ). Assuming the claim, it follows from Lemma 21 that  $\tilde{G}$  fixes the orthogonal complement  $V^\perp$  of the linear span  $V$  of  $\tilde{G}p$ , hence  $f$  is a rotation hypersurface with  $V^\perp$  as axis. Moreover, if  $k \neq n-1$  then  $f|_{Gp}$  must be an embedding by a standard covering map argument. From this and  $f \circ g = \Psi(g) \circ f$  for every  $g \in G$  it follows that any  $g$  in the kernel of  $\Psi$  must fix any point of  $Gp$ . Since  $Gp$  is a principal orbit, this easily implies that  $g = \text{id}$ . Therefore  $\Psi$  is an isomorphism of  $G$  onto  $\tilde{G}$ .

We now prove the claim. By Proposition 19 we have that  $f$  immerses  $Gp$  as an isoparametric submanifold. If condition (i) holds then the first normal spaces of  $f|_{Gp}$  in  $\mathbb{R}^{n+1}$  are one-dimensional. By Theorem 16-(ii) and a well-known result on reduction of codimension of isometric immersions (cf. [Da], Proposition 4.1), we obtain that

$f(Gp) = \tilde{G}(f(p))$  is contained as a hypersurface in some affine hyperplane  $\mathcal{H} \subset \mathbb{R}^{n+1}$ , and hence  $f(Gp)$  must be a round hypersphere of  $\mathcal{H}$  (a circle if  $k = n - 1$ ). Moreover, condition (i) is automatic if  $k = n - 1$ , for in this case the principal orbit of maximal length must be a geodesic. Now, if (iii) holds, let  $Gp$  be a principal orbit such that the position vector of  $f$  is nowhere tangent to  $f(M^n)$  along  $Gp$ . Then any normal vector  $\tilde{\xi}$  to  $f|_{Gp}$  at  $gp \in Gp$  can be written as  $\tilde{\xi} = af(gp) + f_*(gp)\xi_{gp}$ , with  $\xi_{gp}$  normal to  $Gp$  in  $M^n$ . Therefore the shape operator  $A_{\tilde{\xi}}^{f|_{Gp}}$  is a multiple of the identity tensor, hence  $f|_{Gp}$  is umbilical. Since, by (ii), the dimension of  $Gp$  can be assumed to be at least two, the claim is proved also in this case. As for conditions (iv) and (v), the claim follows from Proposition 17-(i) and the last assertion in Proposition 17-(ii), respectively. ■

*Proof of Corollary 5.* By Proposition 19 and part (ii) of Proposition 17 we have that the orbit  $\tilde{G}(f(p)) = f(Gp)$  of  $\tilde{G}$  is an extrinsic product  $\mathbb{S}_1^{n_1-1} \times \dots \times \mathbb{S}_k^{n_k-1}$  of round spheres or circles. In particular, this implies that the orthogonal decomposition  $\mathbb{R}^{n+1} = \bigoplus_{i=0}^k \mathbb{R}^{n_i}$ , where  $\mathbb{R}^{n_i}$  is the linear span of  $\mathbb{S}_i^{n_i-1}$  for  $1 \leq i \leq k$ , is  $\tilde{G}$ -stable, that is,  $\mathbb{R}^{n_i}$  is  $\tilde{G}$ -invariant for  $0 \leq i \leq k$  (cf. [GOT], Lemma 6.2). By [D], Theorem 4 there exist connected Lie subgroups  $G_1, \dots, G_k$  of  $\tilde{G}$  such that  $G_i$  acts on  $\mathbb{R}^{n_i}$  and the action of  $\tilde{G} = G_1 \times \dots \times G_k$  on  $\mathbb{R}^{n+1}$  given by

$$(g_1 \dots g_k)(v_0, v_1, \dots, v_k) = (v_0, g_1 v_1, \dots, g_k v_k)$$

is orbit equivalent to the action of  $\tilde{G}$ . Moreover, writing  $q = f(p) = (q_0, \dots, q_k)$  then  $\tilde{G}(q) = \{q_0\} \times G_1(q_1) \times \dots \times G_k(q_k)$ , and hence  $G_i(q_i)$  is a hypersphere of  $\mathbb{R}^{n_i}$  for  $1 \leq i \leq k$ . The last assertion is clear. ■

## 7 Proof of Theorem 6.

Let  $L^k \times_\rho N^{n-k}$  be a warped product with  $N^{n-k}$  connected and complete and let  $\psi: L^k \times_\rho N^{n-k} \rightarrow U$  be an isometry onto an open dense subset  $U \subset M^n$ . Since  $M^n$  is a compact Riemannian manifold isometrically immersed in Euclidean space as a hypersurface, there exists an open subset  $W \subset M^n$  with strictly positive sectional curvatures. The subset  $U$  being open and dense in  $M^n$ ,  $W \cap U$  is a nonempty open set. Let  $L_1 \times N_1$  be a connected open subset of  $L^k \times N^{n-k}$  that is mapped into  $W \cap U$  by  $\psi$ . Then the sectional curvatures of  $L^k \times_\rho N^{n-k}$  are strictly positive on  $L_1 \times N_1$ .

For a fixed  $x \in L_1$ , choose a unit vector  $X_x \in T_x L_1$ . For each  $y \in N^{n-k}$ , let  $X_{(x,y)}$  be the unique unit horizontal vector in  $T_{(x,y)}(L_1 \times N^{n-k})$  that projects onto  $X_x$  by  $(\pi_1)_*(x, y)$ . Then the sectional curvature of  $L^k \times_\rho N^{n-k}$  along a plane  $\sigma$  spanned by  $X_{(x,y)}$  and any unit vertical vector  $Z_{(x,y)} \in T_{(x,y)}(L_1 \times N^{n-k})$  is given by

$$K(\sigma) = -\text{Hess } \rho(x)(X_x, X_x)/\rho(x).$$

Observe that  $K(\sigma)$  depends neither on  $y$  nor on the vector  $Z_{(x,y)}$ . Since  $K(\sigma) > 0$  if  $y \in N_1$ , the same holds for any  $y \in N^{n-k}$ . In particular,  $L_1 \times_\rho N^{n-k}$  is free of flat points.

If  $n - k \geq 2$ , it follows from [DT], Theorem 16 that  $f \circ \psi$  immerses  $L_1 \times_\rho N^{n-k}$  either as a rotation hypersurface or as the extrinsic product of an Euclidean factor  $\mathbb{R}^{k-1}$  with a cone over a hypersurface of  $\mathbb{S}^{n-k+1}$ . The latter possibility is ruled out by the fact that the sectional curvatures of  $L^k \times_\rho N^{n-k}$  are strictly positive on  $L_1 \times N_1$ . Thus the first possibility holds, and in particular  $f \circ \psi$  immerses each leaf  $\{x\} \times N^{n-k}$ ,  $x \in L_1$ , isometrically onto a round  $(n - k)$ -dimensional sphere. It follows that  $N^{n-k}$  is isometric to a round sphere.

In any case,  $\text{Iso}^0(N^{n-k})$  acts transitively on  $N^{n-k}$  and each  $g \in \text{Iso}^0(N^{n-k})$  induces an isometry  $\bar{g}$  of  $L^k \times_\rho N^{n-k}$  by defining

$$\bar{g}(x, y) = (x, g(y)), \text{ for all } (x, y) \in L^k \times N^{n-k}.$$

The map  $g \in \text{Iso}^0(N^{n-k}) \mapsto \bar{g} \in \text{Iso}(L^k \times_\rho N^{n-k})$  being clearly continuous, its image  $\bar{G}$  is a closed connected subgroup of  $\text{Iso}(L^k \times_\rho N^{n-k})$ . For each  $\bar{g} \in \bar{G}$ , the induced isometry  $\psi \circ \bar{g} \circ \psi^{-1}$  on  $U$  extends uniquely to an isometry of  $M^n$ . The orbits of the induced action of  $\bar{G}$  on  $U$  are the images by  $\psi$  of the leaves  $\{x\} \times N^{n-k}$ ,  $x \in L^k$ , hence are umbilical in  $M^n$ . Moreover, the normal spaces to the (principal) orbits of  $\bar{G}$  on  $U$  are the images by  $\psi_*$  of the horizontal subspaces of  $L^k \times_\rho N^{n-k}$ . Therefore, they define an integrable distribution on  $U$ , whence on the whole regular part of  $M^n$ . Thus, the action of  $\bar{G}$  on  $M^n$  is locally polar with umbilical principal orbits. We conclude from Corollary 4-(iii) that  $f$  is a rotation hypersurface. ■

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